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Diffusion Modeling of the Rayleigh Piston

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A Markov jump process in which a massive labeled particle undergoes random elastic collisions with a thermal bath is investigated. It is found that the behavior of the labeled particle can be divided into three distinct regimes depending on whether its velocity is (1) much less than, (2) on the order of, or (3) much greater than the mean speed of a bath particle. In each regime the jump process can be approximated by a particular continuous-path diffusion process. The first case corresponds to the Ornstein–Uhlenbeck process, while each of the latter can be modeled by a deterministic process with a nonlinear Langevin equation. In addition, in cases (2) and (3), the scaled deviation from the mean velocity can be modeled by a nonstationary diffusion. By scaling the time and letting the mass of the labeled particle become large, a continuous-path diffusion is constructed which approximates the jump process in each regime. Analytic solutions for the transition probability density are provided in each case, and numerical comparisons are made between the mean and variance of the diffusions and the original jump process.

KEY WORDS: Brownian motion; diffusion; nonlinear fluctuations; Rayleigh piston.

1. INTRODUCTION

Consider a labeled particle of mass M subject to random, elastic collisions with a one-dimensional bath of particles, each of mass m. This system was first investigated by Lord Rayleigh in 1891 and has become known as the "Rayleigh piston." Since then, many others have also studied this system.[3] In this paper we shall be interested in modeling the velocity of the labeled particle by a diffusion.

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Under the assumption of elastic collisions, the velocity of the labeled particle (Rayleigh piston) follows a stochastic jump process. It is not feasible to obtain the probability density function (p.d.f.) of velocities at some future time in closed form by solving an associated master equation. However, as $m/M \to 0$ the size of the jumps decreases and an approximation by a diffusion process is possible. For a stationary velocity distribution of the bath particles, it is found that the behavior of the Rayleigh piston can be separated into three distinct regimes corresponding to a velocity (1) much less than, (2) on the order of, or (3) much greater than the root-mean-square (rms) velocity of a bath particle. In each regime the jump process can be approximated by a diffusion. The first case corresponds to the Ornstein–Uhlenbeck process obtained as a solution of the standard linear Langevin equation. In each of the other two cases, the velocity can be modeled by a deterministic process. Here, by a suitable scaling of the deviation of the velocity from its deterministic limit, a related diffusion process is constructed which yields the distribution of velocities about the mean.

In Section 2 we describe in detail the collision process, the master equation, and the infinitesimal moments that will be needed to obtain diffusion approximations for the velocity. We also introduce the related ξ process generated by the scaled deviation of the velocity from its mean, and construct the associated infinitesimal moments. Section 3 is devoted to rigorously constructing the diffusions discussed above by subjecting the original jump process to specific limits. In Section 4, the nonstationary Fokker–Planck equations which govern the diffusions obtained in Section 3 are formally solved. A diffusion approximation valid for high velocities is explicitly solved. In addition comparisons are made between the diffusion approximations and numerical solutions of the master equation for different values of m/M for the specific case where the bath particles have a Maxwell–Boltzmann velocity distribution.

2. THE JUMP PROCESS

Assume that a labeled particle (piston) of mass M and velocity v is moving in a one-dimensional bath of particles of mass m. Collisions between the piston and the bath particles are elastic. After each collision the bath particles are given a new distribution, which evolves until the next collision. The distribution of their positions will be taken as Poisson with density ρ, and their velocities are distributed independently of their positions. Then, by construction, there can be no recollisions. The above assumptions are sufficient to guarantee that the collision times between the Rayleigh piston and the bath particles form a Markov (Poisson) process.

Let f_b be the singlet distribution of velocities and positions of the bath particles. Since velocity and position are independent, we shall only use f_b with one argument, u, denoting velocity. Thus, $f_b(u)/\rho$ is the p.d.f. of the velocity of a given bath particle.

If u, v represent velocities of a bath particle and the piston, respectively, and u', v' are these velocities after an elastic collision, then

$$\begin{aligned} v' &= v + 2\alpha(u - v) \\ u' &= u + 2\alpha(v - u)M/m \end{aligned} \tag{1}$$

where α is defined as $m/(m + M)$.

Given that the piston has velocity v, the probability that there will be a collision in time $(t, t + dt)$ with a bath particle with velocity in the range $(u, u + du)$ is

$$|u - v| f_b(u)\, du\, dt \tag{2}$$

The probability of more than one collision in time dt is $o(dt^2)$. When writing expression (2) we are implicitly assuming that we are dealing with a Markov process since the effects of previous collisions are not taken into account. As previously mentioned, the collision times form a Poisson process with a collision rate, λ, that depends on the velocity of the piston:

$$\lambda(v) = \int |u - v| f_b(u)\, du \tag{3}$$

Let $P(v, t)$ be the density function of the velocity of the piston at time t. From the above assumptions, this velocity will be a Markov process and $P(v, t)$ will satisfy the so-called master equation:

$$\frac{\partial P}{\partial t} = \int |v - u| \left[P(v', t) f_b(u') - P(v, t) f_b(u) \right] du \tag{4}$$

where v' and u' are given in (1). The goal is to obtain $P(v, t)$ in terms of f_b, but this will not be possible using (4) directly. Instead we introduce the infinitesimal moments of the process:

$$K_n(v) = \lim_{\Delta t \to 0} \left\langle (\Delta v)^n \mid v \right\rangle / \Delta t \tag{5}$$

where $\langle \cdots \mid v \rangle$ represents the conditional expectation given that the velocity of the piston is v. Δv represents the change in the velocity of the piston during an interval of length Δt.

Equation (5) can be written as

$$K_n(v) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int (\Delta v)^n W(\Delta v, \Delta t \mid v)\, d(\Delta v) \tag{6}$$

where $W(\Delta v, \Delta t \mid v)$ is the p.d.f. of the transition $v \to v + \Delta v$ during the time interval Δt. Using (2), Eq. (6) can be expressed as

$$K_n(v) = (2\alpha)^n \int |u - v| (u - v)^n f_b(u)\, du \tag{7}$$

Because the velocity of the piston follows a Markov process, $P(v, t)$ will satisfy the Chapman–Kolmogorov equation:

$$P(v, t + \Delta t) = \int P(v - \Delta v, t)\, W(\Delta v, \Delta t \mid v - \Delta v)\, d(\Delta v) \tag{8}$$

Expanding the integrand in (8) about v yields

$$\sum_{n=0}^{\infty} \frac{(-1)^n (\Delta v)^n}{n!} \frac{\partial^n}{\partial v^n} \left[P(v, t)\, W(\Delta v, \Delta t \mid v) \right] \tag{9}$$

Using (6) and (9), Eq. (8) becomes

$$\frac{\partial P}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial v^n} \left[P(v, t) K_n(v) \right] \tag{10}$$

Another series for $\partial P / \partial t$ may be obtained by expanding $P(v', t)$ about $P(v, t)$ in Eq. (4):

$$\frac{\partial P}{\partial t} = \int |v - u| \left[f_b(u') - f_b(u) \right] P(v, t)\, du + \sum_{n=1}^{\infty} \frac{1}{n!} K_n'(v) \frac{\partial^n P(v, t)}{\partial v^n} \tag{11}$$

where

$$\begin{aligned} K_n'(v) &= (2\alpha)^n \int |u - v| (u - v)^n f_b(u')\, du \\ &= (-1)^{n+1} \left(\frac{M + m}{M - m} \right)^{n+2} K_n(v) \end{aligned} \tag{12}$$

Equation (11) frequently occurs in the physics literature.(2,3) The terms are grouped by the order of the derivative of P, rather than powers of α.

As $m/M \to 0$ the velocity of the piston becomes deterministic. To investigate its stochastic component we construct a new process[4] with a scaled variance,

$$\xi(t) = \frac{v(t) - \bar{v}(t)}{\sqrt{\alpha}} \tag{13}$$

where $\bar{v}(t)$ is the mean velocity at time t. If we observe the change in the ξ process over a time Δt then

$$\Delta \xi = (\Delta v - \Delta \bar{v}) / \sqrt{\alpha} \tag{14}$$

[4]This scaling was first introduced in a more general context by Van Kampen.(4)

By the binomial theorem

$$(\Delta\xi)^n = \alpha^{-n/2} \sum_{k=0}^{n} \binom{n}{k} (\Delta v)^{n-k} (-\Delta\bar{v})^k \tag{15}$$

If we let R_n denote the nth infinitesimal moment of the ξ process,

$$R_n = \lim_{\Delta t \to 0} \langle (\Delta\xi)^n \,|\, \xi \rangle / \Delta t \tag{16}$$

then (16) can be expressed using (15) and (6):

$$R_n = \lim_{\Delta t \to 0} \frac{\alpha^{-n/2}}{\Delta t} \sum_{k=0}^{n} \binom{n}{k} (-\Delta\bar{v})^k \int (\Delta v)^{n-k} W(\Delta v, \Delta t \,|\, v)\, d(\Delta v) \tag{17}$$

But since $\Delta\bar{v}$ is of the order Δt, as $\Delta t \to 0$ we obtain

$$R_n = \begin{cases} \dfrac{1}{\sqrt{\alpha}} \left[K_1(\bar{v} + \sqrt{\alpha}\,\xi) - \dfrac{d\bar{v}}{dt} \right], & n = 1 \\ \alpha^{-n/2} K_n(\bar{v} + \sqrt{\alpha}\,\xi), & n > 1 \end{cases} \tag{18}$$

We shall return to this process in Section 3 for a full discussion of its properties.

3. DIFFUSION MODELS

In the previous section we described the behavior of the velocity of the piston as a jump process. To determine the distribution of velocity at a future time we must solve the master equation (4). However, this is only possible numerically. Since we are primarily interested in the case $M \gg m$, we shall try approximating the jump process with a continuous path process, i.e., a Markov diffusion. The probability density of such a process satisfies the Fokker–Planck equation:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial v}\left[b(v,t) P(v,t) \right] + \frac{1}{2} \frac{\partial^2}{\partial v^2} \left[a(v,t) P(v,t) \right] \tag{19}$$

where $b(v,t)$ and $a(v,t)$ are, respectively, the first and second infinitesimal moments of the *continuous* process. An arbitrary way to determine these functions from the jump process is to truncate the series in (10). This leads to the choice $b(v,t) = K_1(v)$ and $a(v,t) = K_2(v)$ with K_1, K_2 given by (7). Truncating the series in (11) yields a different choice. Since the diffusion approximation will only be valid when $m \to 0$ or $M \to \infty$ we would like to verify that the terms $n \geqslant 3$ do indeed vanish while systematically determining correct expressions for a and b. Here we examine possible diffusion approximations when $M\to\infty$. In each case we will indicate how the identical process may be obtained by letting $m\to 0$.

A. $v\sim\sigma_b$**. Let** $M\to\infty$, $\tau=\alpha t$. If we examine the jump process simply by taking the limit $M\to\infty$ we obtain free motion for the piston; the collisions have no effect. Hence we introduce the scaled time $\tau=\alpha t$. By velocity we still mean dx/dt, not $dx/d\tau$. By choosing τ as our time variable we are scaling the observation times so that enough collisions will occur to produce measurable changes in the piston's velocity.

The infinitesimal moments of the scaled velocity process can be written in terms of K_n given by (7):

$$\lim_{\Delta\tau\to 0}\langle(\Delta v)^n\,|\,v\rangle/\Delta\tau=(1/\alpha)K_n(v) \tag{20}$$

The limiting velocity process has infinitesimal moments:

$$\lim_{\alpha\to 0}(1/\alpha)K_n(v)=\begin{cases}2\int|u-v|(u-v)f_b(u)\,du, & n=1\\ 0, & n>1\end{cases} \tag{21}$$

Therefore, the resulting deterministic velocity process, $\bar{v}(\tau)$, is governed by the equation

$$\frac{d\bar{v}}{d\tau}=2\int|u-\bar{v}|(u-\bar{v})f_b(u)\,du \tag{22}$$

Since, from (7), $dK_1/d\bar{v}=-4\alpha\lambda$ the corresponding ξ process has limiting infinitesimal moments

$$\lim_{\alpha\to 0}(1/\alpha)R_n=\begin{cases}-4\xi\lambda(\bar{v}), & n=1\\ 4\int|u-\bar{v}|(u-\bar{v})^2f_b(u)\,du, & n=2\\ 0, & n>2\end{cases} \tag{23}$$

The diffusion that corresponds to the limiting ξ process above is characterized by a Fokker–Planck equation of the form

$$\frac{\partial P}{\partial\tau}=\frac{\partial}{\partial\xi}\left[\xi\cdot B(\tau)P(\xi,\tau)\right]+\frac{1}{2}\frac{\partial^2}{\partial\xi^2}\left[A(\tau)P(\xi,\tau)\right] \tag{24}$$

where A and B depend on τ through $\bar{v}$. The identical process can be obtained by taking $m\to 0$, $\rho\to\infty$ while keeping constant the quantities $(m\rho)$ and σ_b^2, the variance of the velocity of a bath particle.

B. $v\gg\sigma_b$**. Let** $M\to\infty$, $\tau=\alpha t$, $\sigma_b\to 0$. The process defined in Section A is useful for piston velocities $v\sim\sigma_b$. A related process can be constructed which is useful for approximating the jump process when $v\gg\sigma_b$ by letting $f_b\to\rho\delta(u)$ in (21)–(23).

The resulting deterministic velocity process satisfies

$$\frac{d\bar{v}}{dt} = -2\rho\bar{v}|\bar{v}| \tag{25}$$

and the p.d.f. $P(\xi,\tau)$ is the solution of

$$\frac{\partial P}{\partial \tau} = 4\rho|\bar{v}|\frac{\partial}{\partial \xi}(\xi P) + 2|\bar{v}(\tau)^3|\rho\frac{\partial^2}{\partial \xi^2}P \tag{26}$$

This is the diffusion limit of the Rayleigh piston when the bath particles are at rest.[5] Explicit solutions for this process are given in the appendix. Alternatively, it can be obtained from the jump process by taking $m \to 0$, $\rho \to \infty$, $\sigma_b \to 0$ while keeping $(m\rho)$ constant.

C. $v \ll \sigma_b$**. Let** $M \to \infty$, $\tau = \alpha\ t$, $w = v/\sqrt{\alpha}$**.** The favorite model for the physical motion of a massive particle in a fluid is the Ornstein–Uhlenbeck (OU) process. Although this model cannot be obtained from A above because the latter is deterministic, it is a first approximation for A when $v \ll \sigma_b$. [However, note that (24) is the OU process for ξ when $\bar{v} \equiv 0$.]

In order to obtain the OU process as a rigorous limit of the Rayleigh piston it is necessary to scale the piston's velocity in addition to the time. If we define $w = v/\sqrt{\alpha}$ we obtain the following infinitesimal moments for the process w when $\alpha \to 0$:

$$\lim_{\alpha\to 0}\lim_{\Delta\tau\to 0}\frac{1}{\Delta\tau}\left\langle(\Delta w)^n\,|\,w\right\rangle = \lim_{\alpha\to 0}\frac{1}{\alpha^{1+n/2}}K_n\left(w\sqrt{\alpha}\right)$$
$$= \begin{cases} -4w\displaystyle\int|u|f_b\,du, & n=1 \\ 4\displaystyle\int|u^3|f_b\,du, & n=2 \\ 0, & n>2 \end{cases} \tag{27}$$

providing that

$$\int_{-\infty}^{0}u^2 f_b\,du = \int_0^{\infty}u^2 f_b\,du \tag{28}$$

If condition (28) is not satisfied, the first infinitesimal moment diverges and the process is not defined. It should be noted that any f_b which is stationary will also be symmetric in u.[6]

[5] The associated jump process is examined in Ref. 5.

[6] See Ref. 6 for an application of this limit to a Brownian particle interacting with a fluid characterized by a nonuniform temperature.

The probability distribution for w, $P(w,\tau)$, satisfies the familiar version of the Fokker–Planck equation

$$\frac{\partial P}{\partial \tau} = B\frac{\partial}{\partial w}(wP) + \frac{1}{2}A\frac{\partial^2}{\partial w^2}P \tag{29}$$

where here, in contrast with (24) and (26)

$$A = 4\int |u^3| f_b\,du, \qquad B = 4\int |u| f_b\,du \tag{30}$$

are constants independent of τ. Alternatively the OU process can be obtained by letting $m \to 0$, $\rho \to \infty$, $\sigma_b \to \infty$ while keeping $(\rho\sqrt{m})$ and $(\sigma_b\sqrt{m})$ constant.

4. THE SOLUTION

In this section we shall solve the Fokker–Planck equation (24). We first change the time variable from τ (or t) to η:

$$\eta = \int_0^\tau B(y)\,dy \tag{31}$$

Making the change of variable in (24) leads to

$$\frac{\partial P(\xi,\eta)}{\partial \eta} = \frac{\partial}{\partial \xi}[\xi P] + \frac{D(\eta)}{2}\frac{\partial^2 P}{\partial \xi^2} \tag{32}$$

where

$$D(\eta) = A(\tau)/B(\tau) \tag{33}$$

In (33), τ is considered as a function of η through (31). This is possible since B is positive and so (31) has an inverse function $\tau(\eta)$.

Following Ricciardi, it is possible to solve (32) by a transformation to a Wiener process[7] (see Appendix B). This yields a Gaussian probability density with a time-dependent variance:

$$P(\xi,\eta) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\xi^2/2\sigma^2} \tag{34}$$

where σ^2 is the solution of

$$d\sigma^2/d\eta = -2\sigma^2 + D(\eta) \tag{35}$$

so that

$$\sigma^2 = e^{-2\eta}\int_0^\eta D(y)e^{2y}\,dy \tag{36}$$

From these equations we can obtain the approximate distribution of the velocity, rather than ξ. Since

$$v = \bar{v} + \xi\sqrt{\alpha} \tag{37}$$

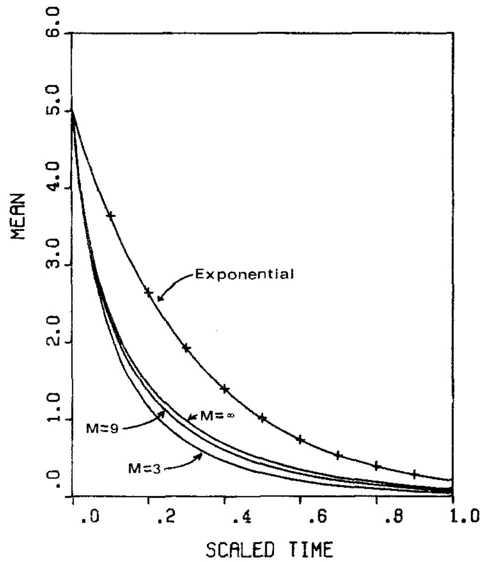

Fig. 1. Mean velocity vs. scaled time for original jump process ($M = 3, 9$), diffusion ($M = \infty$), and Ornstein–Uhlenbeck process (exponential).

we conclude that v will have an approximate Gaussian density with mean $\bar{v}$ and variance $\alpha\sigma^2$. This applies to cases A and B discussed in Section 3.

As a numerical illustration of the degree of approximation resulting from the above approach we consider case A with a Maxwell–Boltzmann distribution of bath particle velocities having $\sigma_b = 1$ and $\rho = 1$. The computations are described in the Appendix. In Fig. 1, $\bar{v}$ for the diffusion process [from (22)] is plotted versus scaled time for the case $v_0 = 5$. It is compared with $\langle v(\tau) \mid v_0 = 5 \rangle$ for the original jump process (with masses $M = 3, 9$) as well as the Ornstein–Uhlenbeck process, for which $\langle v(\tau) \mid v_0 \rangle$ is independent of M in scaled time. The *relatively* slow initial decay of $\langle v(\tau) \mid v_0 \rangle$ for the OU process arises from its failure to account for the rapid rise in collision rate with velocity when $|v| \geqslant \sigma_b$. In Fig. 2, the variance from (36) is compared with the variance from the jump process for the same values of M and v_0. In Fig. 3 the dependence of σ^2 in (36) upon v_0 is illustrated. Notice that only when $v_0 = 0$ do we obtain the exponential increase in the variance as expected from the Ornstein–Uhlenbeck theory. If $v_0 \neq 0$, we obtain a sharp increase in the variance followed by a slow decay to its limiting value ($= 1$ in Fig. 3). It is apparent from Figs. 1 and 2 that the actual jump process rapidly approaches the diffusion model as M becomes

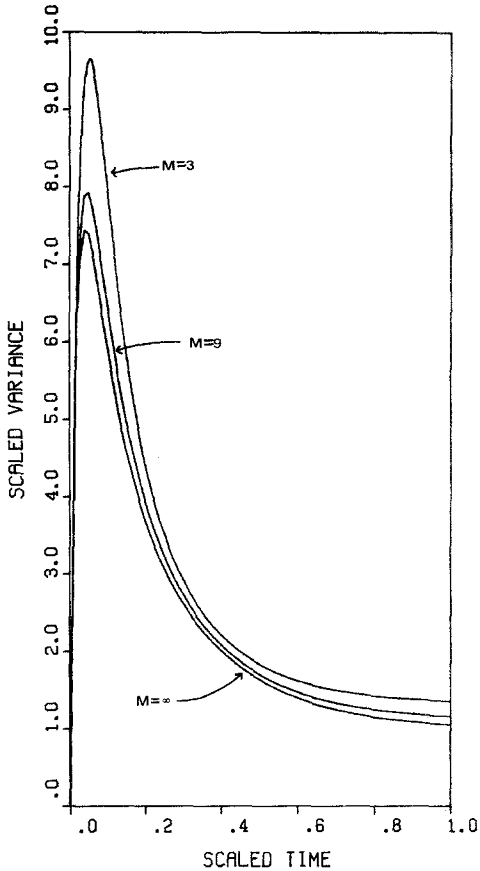

Fig. 2. Scaled variance vs. scaled time for original jump process ($M = 3, 9$) and diffusion ($M = \infty$).

large. In Fig. 4 the variance is plotted for $v_0 = 5$ when the bath is at rest (case B) and compared to the variance curve for this initial velocity from Fig. 3 for a Maxwell–Boltzmann distribution. As anticipated, case (B) yields a good approximation to the piston when $v \gg \sigma_b (= 1$ here). However, in contrast with cases (A) and (C), the piston eventually comes to rest.

5. DISCUSSION

In general, more than one diffusion can be constructed as the limit of a jump process. Each diffusion may approximate the original process in a

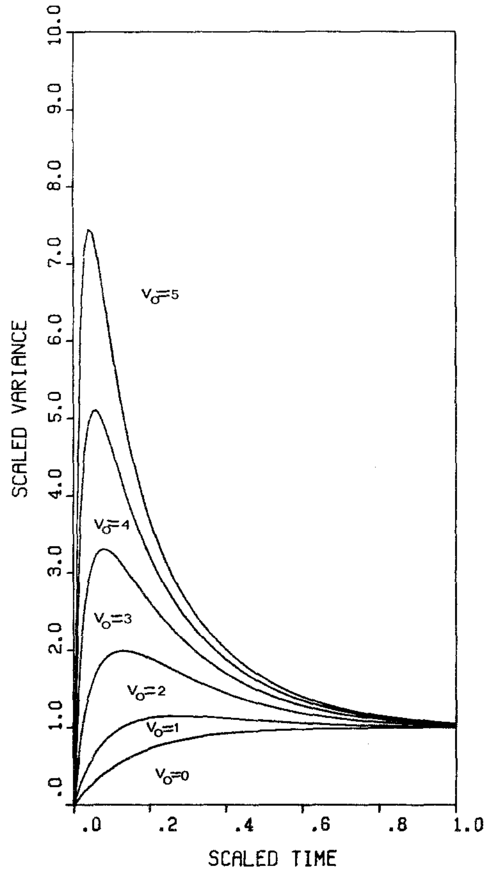

Fig. 3. Scaled variance vs. scaled time for different initial velocities (diffusion).

limited domain. In the case of the Rayleigh piston, the Ornstein–Uhlenbeck process is a useful approximation only when the piston velocity is much less than the mean speed of a bath particle.

It is possible to model the piston over its entire domain by a nonstationary process, case A, characterized by a linear Langevin equation. The Fokker–Planck equation governing this process can be solved by transforming to a Wiener process. The solution for the transition probability density is Gaussian with a time-dependent variance which is related to the nonlinear deterministic equation of motion governing the velocity of the piston. In contrast to the Ornstein–Uhlenbeck process, this process has a

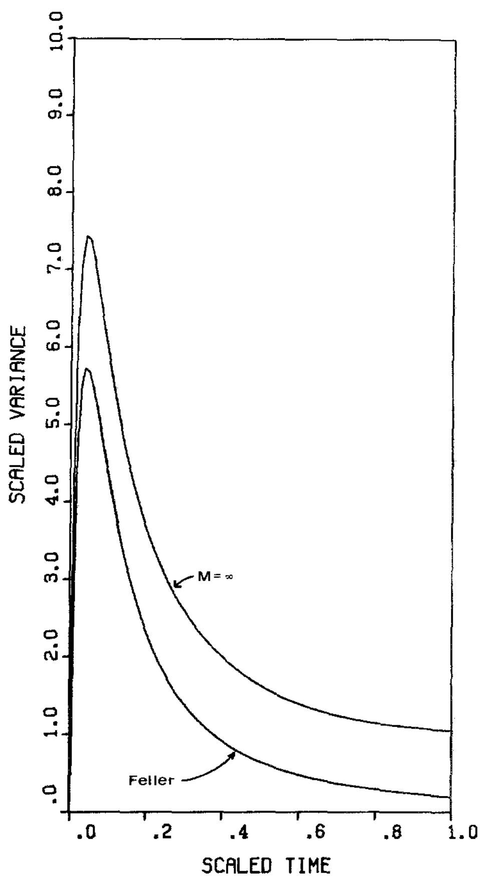

Fig. 4. Scaled variance vs. scaled time for a Maxwell–Boltzmann heat bath ($M = \infty$), and a bath of particles at rest (Feller).

variance which increases rapidly with scaled time, has a single maximum, and then approaches its asymptotic limit from above.

Interesting extensions of the above techniques include Brownian motion in three dimensions, Brownian motion in nonuniform baths,[6] and non-Markovian dynamical models.[8]

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APPENDIX

A. Maxwell–Boltzmann Bath

The collision rate for a Maxwell–Boltzmann velocity distribution of the bath particles with unit density and variance is

$$\begin{aligned}\lambda(v) &= \int |u-v| f_b(u)\,du \\ &= \frac{1}{(2\pi)^{1/2}}\left[\int_v^{\infty} xe^{-x^2/2}\,dx - \int_{-\infty}^{v} xe^{-x^2/2}\,dx - v\int_v^{\infty} e^{-x^2/2}\,dx \right.\\ &\qquad\left. + v\int_{-\infty}^{v} e^{-x^2/2}\,dx\right] \\ &= (2/\pi)^{1/2} e^{-v^2/2} + v\,\mathrm{erf}(v/\sqrt{2})\end{aligned} \tag{38}$$

We also require

$$\begin{aligned}I_n(v) &= \int |u-v|(u-v)^n f_b(u)\,du \\ &= \frac{1}{(2\pi)^{1/2}}\int |x| x^n e^{-(x+v)^2/2}\,dx\end{aligned} \tag{39}$$

But

$$\int |x| x^n e^{-(x+v)^2/2}\,dx = -\left(v + \frac{\partial}{\partial v}\right)\int |x| x^{n-1} e^{-(x+v)^2/2}\,dx$$

Thus, by induction,

$$I_n(v) = (-1)^n\left(v + \frac{\partial}{\partial v}\right)^n \lambda(v) \tag{40}$$

The deterministic velocity process in Fig. 1 was obtained by numerically integrating

$$\frac{d\bar{v}}{d\tau} = 2I_1(\bar{v}) \tag{41}$$

The variance of the ξ process (scaled variance) graphed in Fig. 2 ($M = \infty$) and Fig. 3 is constructed from $A(\tau)$ and $B(\tau)$, where

$$A = 4I_2(\bar{v}), \qquad B = 4\lambda(\bar{v}) \tag{42}$$

B. Nonstationary Fokker–Planck Equation

The solution of

$$\frac{\partial P}{\partial \eta} = \frac{\partial}{\partial \xi}[\xi P] + \frac{1}{2}\frac{\partial^2}{\partial \xi^2}[D(\eta)P] \tag{32}$$

is desired subject to the initial condition $P(\xi, 0) = \delta(\xi)$. Following Ricciardi, introduce the functions x and y,

$$y = y(\eta), \qquad x = x(\xi, \eta) \tag{43}$$

From the condition that the transformation (43) is single valued, x_ξ, $y_\eta > 0$ is required, where the subscripts indicate partial differentiation with respect to the variable indicated. Let f be the p.d.f. for the random variable x. It is related to P by

$$P(\xi, \eta) = f(x, y)x_\xi \tag{44}$$

The prescription is to determine x and y such that f describes a Wiener process,

$$\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2} \tag{45}$$

for which the solution is known,

$$f = \frac{1}{(2\pi y)^{1/2}} e^{-x^2/2y} \tag{46}$$

Substitution of (43) into (32) and comparison with (45) yields the following set of conditions on x and y:

(a) $$y_\eta = Dx_\xi^2$$

(b) $$x_\xi + \xi x_{\xi\xi} - x_{\xi\eta} + \tfrac{1}{2}Dx_{\xi\xi\xi} = 0$$

(c) $$\xi x_\xi^2 + \tfrac{3}{2}Dx_\xi x_{\xi\xi} - x_\xi x_\eta = 0$$

From (a) we have

$$x = \xi(y_\eta / D)^{1/2} + C_1(\eta)$$

From (b) we have

$$(y_\eta / D)^{1/2} - \left(\frac{d}{d\eta}\right)(y_\eta / D)^{1/2} = 0$$

yielding

$$y = C_2 \int_0^\eta D(\eta') e^{2\eta'} \, d\eta' + C_3$$

where C_2 and C_3 are constants.

From (c) we have

$$\frac{dC_1}{d\eta} = 0, \qquad C_1 = \text{const}$$

To satisfy the initial condition on P, take $C_1 = C_3 = 0$. Because of cancellation, P is independent of C_2 and the solution of (32) is

$$P(\xi, \eta) = \left[1/\sigma(2\pi)^{1/2}\right]e^{-\xi^2/2\sigma^2}$$

$$\sigma^2 = e^{-2\eta}\int_0^{\eta} e^{2\eta'}D(\eta')\,d\eta'$$

C. Bath at Rest

For convenience choose the unit of length where $\rho = 1$. The deterministic velocity process $\bar{v}$ is found by integrating (25) to obtain

$$\bar{v}(\tau) = \frac{v_0}{1 + 2\tau|v_0|} \tag{47}$$

with initial velocity v_0.

The variance can be expressed in terms of the scaled time, rather than η. From (31) and (47) we have

$$\eta = 2\ln(v_0/\bar{v})$$

Consequently,

$$\bar{v} = v_0 e^{-\eta/2}, \qquad D(\eta) = v_0^2 e^{-\eta} \tag{48}$$

Using the above results, (36) can be rewritten as

$$\sigma^2 = \frac{4|v_0|^3\tau(1 + \tau|v_0|)}{(1 + 2\tau|v_0|)^4} \tag{49}$$

For any nonzero v_0, Eq. (49) indicates that the variance is a unimodal function of τ. The variance assumes its maximum value of $v_0^2/4$ at $\tau = (\sqrt{2} - 1)/2|v_0|$.

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